Nonlinear Risk of a Bond Portfolio

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Value at risk (VaR) is now widely accepted by practitioners in the Thai market as being an important tool for risk measurement and management. VaR can be defined as the maximum loss that an investor believes, at a certain confidence level, he may incur over the next investment horizon. It can be identified specifically if the investor is willing to state his confidence level and to assume the risk behavior of financial assets.

Asset returns move randomly over time. For the Thai bond and stock markets, research shows that these random returns are far from normal. Yet, my experience suggests it is too expensive for practitioners to deviate from the normality framework for a relatively marginal precision of risk analyses. So, for a practical reason, I will discuss the risk measurement of a bond portfolio with you by assuming normality.

VaR under Normality

Let us suppose the investor has a B-baht investment money. Suppose further that the bond return is normal and the investor sets his confidence level at 99%. Or, he sets his alpha (α) to 0.01 (=1-0.99). The VaR(α) of his bond portfolio can be computed from

$$\text{VaR}(\alpha) = -B \times 2.33\sigma_B.$$

σ_B is the standard deviation of the return on the chosen bond portfolio. I assume a short investment horizon. Hence, the expected return should be small about zero. After all, you will be very lucky if you can trade bonds to earn an average annual return of 25% for the next year and a few years to come. In a very rare case, you may earn up to a 50% return in a certain year. But your scaled daily return is only about 0.0014 (=0.50/365).

It is straightforward to compute the VaR(α) from the given formula, only if you have the estimate of the standard deviation. This statistic is not readily available and must be estimated from the data. Here comes an important pitfall.

On the surface, estimation of the standard deviation should be straightforward. Given the time series of bond return, we can compute that standard deviation even on Excel! Can't we? The answer is a big **NO**.

Here, practitioners must recognize that bonds are aging with time. Each day, we have new but older bonds under the same names, whose time to maturity is shorter and cash payouts are fewer. The law of large number in statistics is not applied for

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these practically changing samples. To proceed, recall that the bond price and return are related systematically with its yield y. This relationship offers an alternative means of computing the bond's VaR(α) from yield.

$$\text{VaR}(\alpha) \;=\; -\,B \times 2.33 \times (\text{Modified Duration} \times \sigma).$$

This result follows the fact that the bond return is approximately proportional with the yield change Δy, with a proportionalty coefficient of minus Modified Duration. σ is the standard deviation of the yield change Δy. This VaR(α) formula is practical because it is quite simple to compute modified duration and traders even have it on their screens.

The estimation of σ can use the time series of bond yield. It is the history of yields on bonds with the same Macualey Duration. For a practical reason, again, I am not very careful with considering the bond portfolio or coupon bond as an equivalent portfolio of zero-coupon bonds and applying the yields appropriate for their respective lives. I also do not push it too far to treat that yield as the spot rate because, I believe, many of you in the industry do not maintain this database. But you can acquire the time series of bond yields from the ThaiBDC.

Just a reminder before I leave this point. Most databases keep bond yields in percentage points per year. When you estimate the standard deviation σ, you must scale the time series to a half-year yield and in decimal points.

Approximate VaR

It should be noted that I get the second VaR(α) formula from the approximation of the bond return by the yield change and Modified Duration. And I further assume that the random change in yield is normal. This approximation is of the first order and can be subjective to a larger error, if the change is large. Yields in the Thai bond market can be very volatile at times. The approximation errors and a volatile market imply that the VaR(α) in the second formula can be seriously understated. It follows that traders who obey that resulting VaR(α) take in too much risk and beyond the allocated risk budget or the investment policy. It is not the traders' faults. This excessive risk could have never been taken, if the VaR(α) had been understood better.

To correct this error, let us look back how we relate the bond return r_B with the yield change Δy.

$$r_B \;=\; (\,-\,\text{Modified Duration} \times \Delta y\,) + \frac{1}{2}(\,\text{Convexity} \times \Delta y^2\,).$$

A simple VaR analysis assumes a small yield change Δy so that Δy^2 disappears. If yield change Δy is large, its square Δy^2 will be significant. It follows that, even if the yield change Δy is normal, the bond return r_B will not be a normal but a *non-central chi-square* variable. This fact refutes the VaR(α) analysis using the $-2.33 \times$ (Modified Duration $\times\, \sigma_{\Delta y}$) formula.

Non-Linear VaR of a Bond Portfolio

A VaR analysis of a bond portfolio may not be simple especially in a volatile market, even though we assume a simplistic normality for the bond yield. This difficulty results from a nonlinear relationship between the bond return and the yield change. A careful VaR analysis must adjust for this non-linearity effect from the bond convexity. I derive the exact form of the bond return r_B for you in the appendix and find that the behavior of r_B can be described by the Modified Duration and Convexity. The non-linear VaR (N-VaR) of the bond portfolio can be obtained, under a zero-mean assumption, from

$$\text{N-VaR}(\alpha) \;=\; B \times (h\chi^2\left(1, \gamma^2; \alpha\right) + k)$$

where $k = -\left(\frac{-\text{Modified Duration}}{2\sqrt{0.5 \times \text{Convexity}}}\right)^2$, $h = \sqrt{0.5 \times \text{Convexity}}\sigma^2$ and $\gamma = \left(\frac{-\text{Modified Duration}}{\text{Convexity} \times \sigma}\right)$. It is not difficult to apply this N-VaR(α) in the risk analysis of a bond portfolio. The variables k, h and γ can be computed using the Modified Duration, Convexity and the standard deviation of the yield change. Although the critical value of the non-central chi-square variable $\chi^2\left(1, \gamma^2; \alpha\right)$ is not reported generally in a statistics textbook, I write a program **Patnaik.xls** on Excel, following the Patnaik procedure, to approximate it. You can download the program from the ThaiBDC's website < www.thaibdc.or.th/patnaik.xls >.

Demonstration

At this point, I should convince you that an extra effort to extend your VaR analysis to a N-VaR framework is important and worthwhile. A simple example should do.

Suppose a trader is interested in a 5-year zero-coupon bond. It is sold today at a 6% yield. The historical 5-year spot rates over the past two and a half years show a standard deviation of 0.00037 (on a half-a-year basis). A simple calculation finds that this bond has a modified duration of 9.7087 and a convexity of 103.6856. The trader assumes normality for the yield change.

Now he has two alternatives to compute the maximum loss possible at a 99% confidence level. First, he can practice the conventional, linear VaR calculation. It gives the maximum loss possible for a one-baht investment of

$$\begin{aligned} \text{VaR}(\alpha) &= -2.33 \times 0.00037 \times 9.7087 \\ &= -0.0008369 \text{ baht.} \end{aligned}$$

Second, if he is aware of the non-linearity effect, he can be more careful and considers the N-VaR. Additional calculation shows the statistics h, k and γ are $7.09e\text{-}6$, -0.4545 and -2.4408, respectively. Setting the centrality parameter to 5.9573 (= -2.4408^2) and the degree of freedom to 1 in the ***Patnaik.xls*** program gives

$\chi^2\left(1, \gamma^2 = 5.9573; \alpha = 0.01\right)$ = 12.945, corresponding with α = 0.01 or the 99% confidence level. The resulting N-VaR(α) is

$$\begin{aligned} \text{N-VaR}(\alpha) &= (7.09e\text{-}6 \times 12.945) - 0.4545 \\ &= -0.4545 \text{ baht.} \end{aligned}$$

Basically, this result says that the trader may lose up to 45.45%, not 0.08% from this venture. What a big difference!

My Final Remarks

In this discussion, I accept that the VaR analysis is very useful and you--investors or bond traders, can use it to measure risk from the investments. But you must be very careful in applying it. A linear VaR that you generally use ignores the effect of a large movement in yields. This simplistic treatment enables the analysis to relate the risk of bond return linearly with the risk of the yield change. However, the yield movement can be large and the non-linearity effect of the second order can be significant. In a volatile market, I suggest you to apply the non-linear VaR. It is not difficult to compute one. And, even though it gives you an uncomfortably large number, it tells you exactly what kind of risk you bring home to spend the night with.

APPENDIX: A Bond Return as A Non-Central Chi-Squared Variable

First, note that the call price is a function of the stock price.

$$B = B(y).$$

From a second-order Taylor series approximation, one has

$$dB = B_y\, dy + 0.5\, B_{yy}\, (dy)^2,$$

where B_y is the dollar duration and B_{yy} is the dollar convexity, leading to

$$\begin{aligned} \frac{dB}{B} &= \frac{B_y}{B} dy + 0.5\, \frac{B_{yy}}{B}\, (dy)^2 \\ &= -D\, dy + 0.5\, C\, (dy)^2. \end{aligned}$$

$\frac{B_y}{B} = -D$ and $\frac{B_{yy}}{B} = C$ are the minus modified duration and the convexity, respectively. Note that $\frac{dB}{B}$ is the bond return r_B. It is assumed that the yield change dy is distributed normally with a mean μ and a standard deviation σ^2. Consider the first term on the RHS.

$$-D\, dy = -D\, dy - D\mu + D\mu$$

$$= \quad - D\,(dy - \mu) - D\mu.$$

The second term on the RHS is

$$0.5\,C\,(dy)^2 = 0.5\,C\,\{dy^2 - 2\,dy\,\mu + \mu^2 + 2\,dy\,\mu - \mu^2\}$$

$$= 0.5\,C\,(dy - \mu)^2 + 0.5\,C\,(2dy\,\mu - \mu^2)$$

$$= 0.5\,C\,(dy - \mu)^2 + C\mu\,(dy - \mu) + 0.5\,C\mu^2.$$

Hence,

$$r_B = - D\,(dy - \mu) - D\mu$$

$$+ 0.5\,C(dy - \mu)^2 + C\mu\,(dy - \mu) + 0.5\,C\mu^2$$

$$= (0.5C\mu^2 - D\mu) + (C\mu - D)(dy - \mu) + 0.5\,C(dy - \mu)^2$$

$$= (0.5C\mu^2 - D\mu) + \left\{ \left(\sqrt{0.5C}\,(dy - \mu)\right)^2 \right.$$

$$\left. + 2\sqrt{0.5C}\,(dy - \mu)\frac{(C\mu - D)}{2\sqrt{0.5C}} + \left(\frac{C\mu - D}{2\sqrt{0.5C}}\right)^2 \right\}$$

$$- \left(\frac{C\mu - D}{2\sqrt{0.5C}}\right)^2$$

$$= (0.5C\mu^2 - D\mu)$$

$$+ \left\{\sqrt{0.5C}\,(dy - \mu) + \left(\frac{C\mu - D}{2\sqrt{0.5C}}\right)\right\}^2 - \left(\frac{C\mu - D}{2\sqrt{0.5C}}\right)^2$$

$$= (0.5C\mu^2 - D\mu) - \left(\frac{C\mu - D}{2\sqrt{0.5C}}\right)^2$$

$$+ 0.5C\sigma^2\left\{\frac{dy - \mu}{\sigma} + \left(\frac{C\mu - D}{C\sigma}\right)\right\}^2.$$

Now, define

$$r_B = k + h\{\gamma + \tilde{z}\}^2,$$

where

$$k = (0.5C\mu^2 - D\mu) - \left(\frac{C\mu - D}{2\sqrt{0.5C}}\right)^2$$

$$h = 0.5C\sigma^2$$

$$\gamma = \left(\frac{C\mu - D}{C\sigma}\right)$$

and $\tilde{z} = \dfrac{dy - \mu}{\sigma}$ is a standard normal variable[2]. For this reason,

$$\frac{r_B - k}{h} = \{\gamma + \tilde{z}\}^2$$

is a non-central chi-square variable $\chi^2(1, \gamma^2)$ with 1 degree of freedom and a non-centrality parameter $\gamma^2 = \left(\dfrac{C\mu - D}{C\sigma}\right)^2$. The ***critical return*** r_B^* at a (1-α) confidence level can be found by

$$r_B^*(\alpha) = h\chi^2(1, \gamma^2; \alpha) + k$$

where $\chi^2(1, \gamma^2; \alpha)$ can be approximated by the Patnaik procedure. See the file "**Patnaik.xls**".

[2] A standard normal variable $\tilde{z}$ has a zero mean and a unit variance. $\tilde{z}^2$ is distributed as a central chi-square variable with one degree of freedom. $(\tilde{z} + c)^2$ is a non-central chi-square variable with one degree of freedom with a non-centrality parameter c^2. The constant c can be positive or negative.